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Critical points of glueball superpotentials and equilibria of integrable systems

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Abstract: We compare the matrix model and integrable system approaches to calculating the exact vacuum structure of general $\mathcal{N}=1$ deformations of either the basic $\mathcal{N}=2$ theory or its generalization with a massive adjoint hypermultiplet, the $\mathcal{N}=2^*$ theory. We show that there is a one-to-one correspondence between arbitrary critical points of the Dijkgraaf-Vafa glueball superpotential and equilibrium configurations of the associated integrable system. The latter being either the periodic Toda chain, for $\mathcal{N}=2$, or the elliptic Calogero-Moser system, for $\mathcal{N}=2^*$. We show in both cases that the glueball superpotential at the crtical point equals the associated hamiltonian. Our discussion includes an analysis of the vacuum structure of the $\mathcal{N}=1^*$ theory with an arbitrary tree-level superpotential for one of the adjoint chiral fields.

Keywords: Nonperturbative Effects, Supersymmetric Effective Theories, Integrable Equations in Physics, Matrix Models.

Contents

1. Introduction

There are a number of ways of investigating the vacuum structure of an $\mathcal{N} = 1$ supersymmetric gauge theory. In this paper we have in mind two such theories, the basic $\mathcal{N} = 2$ theory and also its generalization with a massive adjoint hypermultiplet (known as the $\mathcal{N} = 2^*$ theory), both with gauge group $\mathrm{U}(N)$, deformed by an arbitrary superpotential for the adjoint chiral multiplet. We will focus on two techniques, based on integrable systems [1]–[4] and on matrix models [5]–[8]. The other closely related approach involves formulating the problem in terms of Seiberg-Witten theory [9, 10].

In the integrable system approach the vacua are determined by the extrema of the conserved quantity associated to the $\mathcal{N} = 1$ deformation. In other words, the vacua correspond to the equilibria of the associated flow. On the matrix model side the vacua are determined by the Dijkgraaf-Vafa glueball superpotential. In this note we show that there is a one-to-one correspondence between these approaches and at an equilibrium point the spectral curve of the integrable system is equal to the auxiliary Riemann surface of the matrix model, extending the results of [21] for the massive vacua of the $\mathcal{N} = 1^*$ theory. For these special vacua, the equilibria are stationary with respect to all of the flows of the integrable system signalled by the fact that the spectral curve degenerates all the way to genus one. For a general vacuum this will not be the case and the spectral curve will only partially degenerate. In order to complete the proof we show that at a critical point the value of the glueball superpotential equals the hamiltonian in the integrable system.

The relation between integrable systems and matrix models has also been investigated in [11] for the basic $\mathcal{N} = 2$ case. As we shall see, our approach is rather different, although ultimately must be related. Our approach has some overlap with the recent paper [12] which considers, in the context of the $\mathcal{N} = 2^*$ theory, the relation between the Riemann surfaces of the matrix model and the Seiberg-Witten curve for the $\mathcal{N} = 2^*$ theory constructed by Donagi and Witten [13].

2. The basic $\mathcal{N} = 2$ case

First we describe the integrable system approach (for a general review of integrable systems see [14]). The Coulomb branch of the $\mathcal{N} = 2$ theory with gauge group $\mathrm{U}(N)$ is identified with the moduli space $\mathcal{M}_{\rm int}$ of the spectral curve $\Sigma_{\rm int}$ of the periodic Toda chain, or $A^{(1)}_{N-1}$ affine Toda, integral system. The spectral curve is the Riemann surface defined by the equation

$$F(x, z) = \det\big(x\mathbf{1} + L(z)\big) = 0\,, \tag{2.1}$$

where $L(z)$ is the $N \times N$ Lax matrix which depends on the canonical variables $\{p_i, q_i\}$ via

$$L(z) = \begin{pmatrix} p_1 & e^{q_1-q_2} & 0 & \cdots & z \\ 1 & p_2 & e^{q_2-q_3} & \cdots & 0 \\ 0 & 1 & p_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z^{-1}\Lambda^{2N}e^{q_N-q_1} & 0 & \cdots & 1 & p_N \end{pmatrix}. \tag{2.2}$$

In the present context, the canonical variables are considered to be complex. The N independent flows of the system can be written in Lax form

$$\partial_{t_i} L(z) = [M_i, L(z)]\,, \tag{2.3}$$

where M_i are $N \times N$ matrices. It is then clear that quantities of the form

$$H = \mathrm{Tr}\,\tilde{W}(L(z))\,, \tag{2.4}$$

for an arbitrary polynomial function $\tilde{W}(x)$, are conserved. In particular, there are N independent conserved quantities that can be taken to be

$$H_i = \mathrm{Tr}\,L^i(z)\,, \qquad i = 1, \ldots, N\,. \tag{2.5}$$

Note that the $\{H_i\}$ are independent of z up to a additive constant for $i = N$. Any quantity of the form (2.4) is some function of the $\{H_i\}$ and will generate a flow ∂_t that is some linear combination of the basic flows ∂_{t_i}. Note that the basic conserved quantities $\{H_i\}$ play the rôle of coordinates on $\mathcal{M}_{\rm int}$, the moduli space of $\Sigma_{\rm int}$.

The curve (2.1) can be written more explicitly as

$$F(x, z) = \prod_{i=1}^{N}(x - \tilde{a}_i) + z + \Lambda^{2N}z^{-1} = 0\,, \tag{2.6}$$

where the $\tilde{a}_i = \tilde{a}_i(H_j)$ are an alternative set of coordinates on $\mathcal{M}_{\rm int}$. The curve can be written in hyperelliptic form by defining

$$y = 2z + \prod_{i=1}^{N}(x - \tilde{a}_i) \tag{2.7}$$

in terms of which it takes the form

$$y^2 = \prod_{i=1}^{N} (x - \tilde{a}_i)^2 - 4\Lambda^{2N} \,. \tag{2.8}$$

This has the form of a double-sheeted cover of the complex x-plane with N cuts joining the sheets. Consequently $\Sigma_{\rm int}$ is a genus $N-1$ Riemann surface.

The centre-of-mass motion associated to $\sum_i p_i$ and $\sum_i q_i$ is trivial and we can ignore it. The remaining $N-1$ conjugate angle variables of the integrable system are naturally identified with a point ψ_j, $j = 1\ldots, N-1$, in the jacobian torus of the Riemann surface $\Sigma_{\rm int}$. The jacobian torus is defined as follows (for reference on Riemann surfaces see [15]). First we choose a canonical set of 1-cycles on $\Sigma_{\rm int}$, (A_j, B_j), with intersections $A_j \cdot A_k = B_j \cdot B_k = 0$, $A_j \cdot B_k = \delta_{jk}$, $j,k = 1,\ldots,N-1$. Let ω_j be the associated set of $N-1$ holomorphic 1-forms (abelian differentials of the 1st kind) normalized so that $\oint_{A_j} \omega_k = \delta_{jk}$. The period matrix of $\Sigma_{\rm int}$ is the $N-1 \times N-1$ matrix with elements

$$\tau_{jk} = \oint_{B_j} \omega_k \,. \tag{2.9}$$

The jacobian torus consists of points $\psi_j \in \mathbb{C}^{N-1}$ with the identifications

$$\psi_j \sim \psi_j + n_j + \tau_{jk} m_k \,, \qquad n_j, m_k \in \mathbb{Z} \,. \tag{2.10}$$

We are interested in the flow generated by the conserved quantity of the form H in (2.4) where $\tilde{W}(x)$ is some polynomial of degree n. Notice that $\tilde{W}(x)$ itself will depend on z in such a way that H is z-independent. For any choice of hamiltonian H, the associated dynamics is linear in the jacobian. In other words, for each hamiltonian H, there is a linear flow

$$\psi_j(t) = \varpi_j t + \psi_j(0) \,, \tag{2.11}$$

where ϖ_j are the angular velocities which just depends on $\Sigma_{\rm int}$. It is determined by the unique meromorphic 1-form Ω on $\Sigma_{\rm int}$ via

$$\varpi_j = \oint_{B_j} \Omega \tag{2.12}$$

normalized by

$$\oint_{A_j} \Omega = 0 \tag{2.13}$$

and which is holomorphic on $\Sigma_{\rm int} - P_\pm$, where $P_\pm$ are the two points at $x = \infty$ on the upper and lower sheet. The singularities at $P_\pm$ are specified uniquely by the conserved quantity H in the following way. For each polynomial $\tilde{W}(x)$ in (2.4) there is a unique polynomial $W(x)$ of the same order, for which

$$\lim_{P \to P_\pm} \Omega(P) = \pm d\left(W'(x) + \mathcal{O}\left(\frac{1}{x}\right)\right) . \tag{2.14}$$

Since $W(x)$ is some fixed polynomial and doesn't depend on z by choice, it must be the case that $\tilde{W}(x)$ depends on z in such a way that the associated hamiltonian H in (2.4) is z-independent. We will make the relation between $W(x)$ and $\tilde{W}(x)$ more explicit later.

We now consider how to use the integrable system to find the vacuum structure of the deformed $\mathcal{N} = 2$ theory. In the field theory, the deformation is achieved by adding a tree-level superpotential

$$\frac{1}{g_{YM}^2} \mathrm{Tr}\, W(\Phi) \tag{2.15}$$

to the bare lagrangian of the $\mathcal{N} = 2$ theory. Here, Φ is the $\mathcal{N} = 1$ chiral superfield of the $\mathcal{N} = 2$ theory. In the low-energy effective theory the deformation turns on a potential on the Coulomb branch. One can approach the vacuum problem directly in the four dimensional theory by using Seiberg-Witten theory. The Seiberg-Witten curve is the spectral curve of the integrable system and the Coulomb branch is parameterized by the conserved quantities $\{H_i\}$. Vacua which survive breaking to $\mathcal{N} = 1$, correspond to special points on the Coulomb branch where a number dyons become massless and condense after breaking to $\mathcal{N} = 1$ [9, 10]. However, in order to relate the problem directly to the integrable system it turns out to be more useful to compactify the theory on a circle to three dimensions [1, 2, 3]. This is because the dimension of the Coulomb branch is then doubled by the addition of the Wilson lines and dual photons of the unbroken $\mathrm{U}(1)^N$ gauge group. The resulting Coulomb branch of the three-dimensional theory is identified with the (complexified) total phase space of the integrable system where we have not only the hamiltonians but also the conjugate angle variables.

It has been shown [1, 2, 3] in the context of the $\mathcal{N} = 2^*$ theory, and its quiver generalizations, that the effect of the $\mathcal{N} = 1$ deformation can be captured—including all the quantum corrections—in the three-dimensional theory by taking the superpotential on the Coulomb branch of the three-dimensional theory to be the conserved quantity H in (2.4). We expect these facts to be true in the basic $\mathcal{N} = 2$ theory as well, since this theory can be obtained by taking the large mass limit of the $\mathcal{N} = 2^*$ theory. This philosophy has also been advocated in [11].

Putting aside the usual caveats, supersymmetric vacua are critical points of the superpotential and are therefore points in the complexified phase space which are stationary under the flow ∂_t generated by the hamiltonian H. The value of the superpotential, i.e. H, at the critical point is then valid in the four-dimensional limit. So from the integrable system point-of-view, as first pointed out in [2], we have to identify equilibrium points of the complexified integrable system under the flow ∂_t generated by H. This means that the vacua correspond to points in the moduli space where the vector of angular velocities ϖ_j vanishes; in other words, at the equilibrium point, due to (2.12) and (2.13), it follows that

$$\oint_{A_j} \Omega = \oint_{B_j} \Omega = 0\,. \tag{2.16}$$

This implies that there exists a meromorphic function G on Σ_{int} such that

$$\Omega = dG \tag{2.17}$$

with singularities at $P_\pm = P$ of the form

$$\lim_{P \to P_\pm} G(P) = \pm W'(x) + \mathcal{O}\left(\frac{1}{x}\right). \tag{2.18}$$

So to summarize: the supersymmetric vacua correspond to points in the moduli space $\mathcal{M}_{\text{int}}$ where there exists a meromorphic function G on Σ_{int} with a particular pole structure (2.18) at $P_\pm$. Notice from G we can form the meromorphic function $G + W'(x)$ whose only singularity is a pole at P_+. Now we can bring the Riemann-Roch Theorem to bear on the question of the existence of G. First of all, P_+ is not a Weierstrass point[1] of Σ_{int}, since the latter is hyperelliptic and the Weierstrass points are located at the $2N$ branch points. Therefore, the existence of a meromorphic function with a pole of order n at P_+ requires that Σ_{int} degenerates to a surface of genus $< n$. In other words, in the hyperelliptic form (2.8), it must be that

$$y^2 = \tilde{y}^2 \prod_{i=1}^{N-n} (x - r_i)^2 \,, \tag{2.19}$$

where

$$\tilde{y}^2 = \prod_{i=1}^{2n} (x - s_i) \tag{2.20}$$

describes a surface of genus $< n$. In particular, the meromorphic function G is identified with $\tilde{y}$. The pole structure of G (2.18) then fixes $\tilde{y}$ uniquely to be

$$\tilde{y}^2 = W'(x)^2 + f(x)\,, \tag{2.21}$$

where $f(x)$ is some fixed polynomial of degree $n - 1$. Note that at the equilibrium point where Σ_{mm} degenerates to a surface of lower genus $n-1$, the jacobian must also degenerate. This signals the fact that the equilibrium point of ∂_t is also stationary with respect to other flows. In general if the reduced curve has genus g then there will be $N - g - 1$ stationary flows.

These facts dovetail completely with the matrix model approach [5] and, for that matter, the Seiberg-Witten theory approach [9, 10]. Without reviewing the matrix model approach, we simply note that for an $\mathcal{N} = 1$ deformation described by $W(x)$ in (2.15), the solution of the matrix model involves a hyperelliptic Riemann surface of the form

$$\tilde{y}^2 = W'(x)^2 + f(x)\,, \tag{2.22}$$

where $f(x)$ is a polynomial of degree $n - 1$. This curve is manifestly identical with the spectral curve of the integrable system at the equilibrium point under the flow associated to H.

To complete the picture, we now prove that the value of the hamiltonian H that generates the stationary flow is stationary is equal to the critical value of the Dijkgraaf-Vafa superpotential. The fact that we can find the hamiltonian that generates the flow

[1] The Weierstrass points $\{P_i\}$ are the finite set of points on a Riemann surface for which for each P_i there exists a non-trivial meromorphic function with a singularity only at P_i with an order less than or equal to the genus.

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described by the abelian differential Ω rests on our knowledge of the conjugate action variables the the angle variables [25, 26, 27] (the elliptic Calogero-Moser case was first considered in [28]. The conjugate angle variables a_j, $j = 1\ldots, N-1$, are given by integrals of a certain abelian differential of the 3rd kind λ:

$$a_j = \oint_{A_j} \lambda \,. \tag{2.23}$$

λ is precisely the "Seiberg-Witten" 1-form for the U(N) theory, as deduced in [29, 30]. The defining property of λ is that

$$\oint_{A_k} \frac{\partial}{\partial a_j}\lambda = \delta_{jk} \,. \tag{2.24}$$

We will take

$$\lambda = -\log(y+P)dx \,, \tag{2.25}$$

which is the appropriate form when the derivative in (2.24) is understood to be taken at fixed x. This is necessary in order that the a_j-derivative can be pulled out of the x-integral below. Defining

$$\pi_j = \frac{\partial}{\partial a_j}\lambda \,, \tag{2.26}$$

we see that π_j is an abelian differential of the 3rd kind normalized by

$$\oint_{A_j} \pi_k = \delta_{jk} \tag{2.27}$$

and with simple poles at $P_\pm$. We now apply Riemann's bilinear relation to the abelian differentials π_j and Ω:

$$\frac{1}{2\pi i}\sum_{k=1}^{N-1} \oint_{A_k}\Omega \oint_{B_k}\pi_j - \oint_{B_k}\Omega \oint_{A_k}\pi_j = \mathrm{Res}_{P_+}(W'(x)\pi_j) - \mathrm{Res}_{P_-}(W'(x)\pi_j)\,, \tag{2.28}$$

where we have used the asymptotic forms for Ω at $P_\pm$ in (2.14). Now we use the normalization conditions (2.13) and (2.27), along with the definition of the angular velocities (2.12) and the fact that the contribution from P_- is minus that at P_+, to arrive at

$$\begin{aligned}\frac{1}{4\pi i}\varpi_j &= -\mathrm{Res}_{P_+}(W'(x)\pi_j)\\ &= \frac{\partial}{\partial a_j}\mathrm{Res}_{P_+}(W'(x)\log(y(x)+P(x))dx)\\ &= -\frac{\partial}{\partial a_j}\mathrm{Res}_{P_+}\left(W(x)\frac{P'(x)}{y(x)}dx\right)\end{aligned} \tag{2.29}$$

where in the last line we integrated by parts. Now since a_j is canonically conjugate to the angle ψ_i, this means that the hamiltonian which generates the flow associated to Ω is

$$H = -\mathrm{Res}_{P_+}\left(W(x)\frac{P'(x)}{y(x)}dx\right)\,. \tag{2.30}$$

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It is straightforward to show that if $W(x)$ is a polynomial of order less than N then

$$H = \operatorname{Tr} W(L(z)) \,, \tag{2.31}$$

where $L(z)$ is the Lax matrix. So in this case we can identify $\tilde{W}(x)$ and $W(x)$. If $W(x)$ has an order $\geq N$ then $\tilde{W}(x)$ will be z-dependent. For instance, for $W(x) = x^N$

$$\tilde{W}(x) = x^N + (-1)^N \left(\frac{z + \Lambda^{2N}}{z} \right) . \tag{2.32}$$

What is particularly nice about the result (2.30) is that at a critical point it agrees precisely with the critical value of the Dijkgraaf-Vafa superpotential. In order to see that, one simply deforms the contour around infinity to a sum of contours around the cuts on the degenerated surface. We then identify the differential $P' \, dx/y$ with the resolvent of the field Φ defined in [31, 32]

$$T(x) = \operatorname{Tr} \frac{dx}{x - \Phi} \,. \tag{2.33}$$

The critical value of the superpotential is then

$$\int_{\text{cuts}} W(x) T(x) \tag{2.34}$$

which is equal to (2.30). Hence we find perfect argument between the two distinct approaches for calculating the vacuum structure.

3. The $\mathcal{N} = 2^*$ case

We now apply the same philosophy established in the basic $\mathcal{N} = 2$ case to $\mathcal{N} = 1$ deformations of the $\mathcal{N} = 2^*$ theory.

The Coulomb branch of the U(N) $\mathcal{N} = 2^*$ theory is identified with the moduli space of the spectral curve Σ_{int} of the N-body elliptic Calogero-Moser integrable system [13, 16]:

$$F(v, z) = \det\big(v \mathbf{1} + L(z)\big) = 0 \,, \tag{3.1}$$

where the $N \times N$ Lax matrix $L(z)$ has components

$$L_{ij}(z) = p_i \delta_{ij} + m(1 - \delta_{ij}) \frac{\sigma(z - q_i + q_j)}{\sigma(z)\sigma(q_i - q_j)} e^{\zeta(z)(q_i - q_j)} \,. \tag{3.2}$$

We denote the N-dimensional moduli space of the curve by $\mathcal{M}_{\text{int}}$. The quantity v is a meromorphic function on Σ_{int} with N simple poles at the pre-images of $z = 0$ with residues

$$m\big(N - 1, -1, -1, \ldots, -1\big) \,. \tag{3.3}$$

Notice that one of the points, which we denote P_0, is distinguished by the fact that the residue is $m(N-1)$. In the type-IIA brane construction of Witten [17] P_0 is the position of the NS5-brane. From a field theory perspective, m is the mass of the adjoint hypermultiplet and τ is the bare complexified gauge coupling.

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$F(v,z)$ is a polynomial of degree N in v whose coefficients are elliptic functions on the torus E_τ with complex structure τ:

$$F(v,z) = \sum_{i=0}^{N} f_i(z) v^i \,, \tag{3.4}$$

where

$$f_i(z+2\pi i) = f_i(z+2\pi i\tau) = f_i(z)\,. \tag{3.5}$$

The spectral curve $\Sigma_{\rm int}$ describes an N-sheeted cover of the torus E_τ joined by branch cuts to make a higher genus surface. The number of branch cuts corresponds to the number of zeros of $\partial_v F(v,z)$. Since the latter is a meromorphic function on $\Sigma_{\rm int}$ the degree of its zeros is equal to the degree of its poles. It follows from (3.3) that $\partial_v F(v,z)$ has $N-1$ simple poles and a pole of order $N-1$ at the N pre-images of $z=0$. Hence, there are $2(N-1)$ branch cuts and the Riemann-Hurwitz Theorem gives the genus of $\Sigma_{\rm int}$ as N. We can view the surface as N copies of the torus E_τ glued together by $N-1$ tubes to make a genus N surface.

Any quantity of the form

$$H = \mathrm{Tr}\, \tilde{W}(L(z))\,, \tag{3.6}$$

for a polynomial function $\tilde{W}(x)$, will be conserved. A basis for the hamiltonians, and so a set of coordinates for $\mathcal{M}_{\rm int}$, is provided by

$$H_i = \mathrm{Tr}\, L^i(z)\,, \qquad i=1,\ldots,N\,, \tag{3.7}$$

where z takes a fixed value $\neq 0$. The N conjugate angle variables are, as before, associated with a point $\psi)j$ in jacobian of $\Sigma_{\rm int}$. The dynamics is linear in the jacobian [14, 18, 19] and for the flow corresponding to an arbitrary hamiltonian H in (3.6), we have

$$\psi_j(t) = \varpi_j t + \psi_j(0)\,, \tag{3.8}$$

where ϖ_j is the vector of angular velocities associated to H. This quantity is determined by the unique meromorphic 1-form Ω on $\Sigma_{\rm int}$ with

$$\oint_{A_j} \Omega = 0\,, \qquad \varpi_j = \oint_{B_j} \Omega \tag{3.9}$$

such that it is holomorphic on $\Sigma_{\rm int} - P_0$ with a given singularity at P_0 determined by H. Let x be a coordinate in the neighbourhood of P_0 with $x(P_0)=\infty$, then the singularity of Ω has the form

$$\lim_{P\to P_0} \Omega(P) = d\big(U(x) + \mathcal{O}(1/x)\big)\,, \tag{3.10}$$

where $U(x)$ is a polynomial in x of the same degree as $\tilde{W}(x)$. In our philosophy, $U(x)$ will be fixed uniquely by the $\mathcal{N}=1$ deformation, as we shall see later. This then fixes the hamiltonian H and hence the function $\tilde{W}(x)$. In particular, since $U(x)$ is by choice z-independent, $\tilde{W}(x)$ must be z-dependent in such a way that H in (3.6) is independent of z.

Following the logic of the last section, we now use the structure of the integrable system to find the vacuum structure of the $\mathcal{N}=2^*$ theory deformed to $\mathcal{N}=1^*$. The deformation is achieved by adding a tree-level superpotential

$$\frac{1}{g_{YM}^2}\mathrm{Tr}\, W(\Phi) \tag{3.11}$$

to the bare lagrangian of the $\mathcal{N}=2^*$ theory. Here Φ is massless adjoint chiral superfield of the $\mathcal{N}=2^*$ theory. As before we compactify to three dimensions and identify the Coulomb branch of the theory with the (complexified) total phase space of the integrable system.

The deformation (3.11) is captured exactly by the superpotential on the Coulomb branch of three-dimensional theory which is equal to the hamiltonian H whose flow was described above. Note that in general the two polynomial functions $W(x)$ and $\tilde{W}(x)$ are not equal, however, they are of the same order and are uniquely related to one another—at least implicitly—as we shall see later. Supersymmetric vacua are critical points of the superpotential and are therefore points in the complexified phase space which are stationary under the flow ∂_t generated by the hamiltonian H. Consequently, at these point the associated angular velocities must vanish; hence,

$$\oint_{A_j}\Omega=\oint_{B_j}\Omega=0\,. \tag{3.12}$$

This implies that there exists a meromorphic function G on $\Sigma_{\rm int}$ such that

$$\Omega=dG\,, \tag{3.13}$$

with, from (3.10), a singularity at P_0, of the form

$$\lim_{P\to P_0}G(P)=U(x)+\mathcal{O}\left(\frac{1}{x}\right). \tag{3.14}$$

If $W(x)$ is a polynomial of degree n then so are both $\tilde{W}(x)$ and $U(x)$. Generically, according to the Riemann-Roch Theorem, for G to exist, $\Sigma_{\rm int}$ must degenerate to a surface of genus $n-1$. However, if P_0 happens to be a Weierstrass point of $\Sigma_{\rm int}$ then there may exist additional vacua where $\Sigma_{\rm int}$ has genus $>n-1$. In particular, n would have to be in the "non-gap" sequence at P_0. Unlike in the hyperelliptic situation described in the last section these exceptional cases do indeed occur. An example occurs in the U(3) theory with the simplest quadratic deformation $W(x)=x^2$. In this case, there is a vacuum described in [13] where the surface degenerates from genus 3 to 2 but not all the way to genus 1 as would be required if P_0 was a generic point on the surface. In particular, for the quadratic deformation, as was first pointed out in [2] all the vacua must be described by a degeneration to a hyperelliptic surface since it is only on these surfaces that there exists a meromorphic function with a double pole. In particular, P_0, being a Weierstrass point, must lie at one of the branch points in the two-sheeted representation. We will leave a more in-depth discussion of these exceptional cases to future work.

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Before we move on to describe the matrix model approach to the same problem, we note that we can map the curve $\Sigma_{\rm int}$ into the complex plane parameterized by x via

$$x = iv(z) + im\left(\zeta(z) - \frac{\zeta(\pi i)z}{\pi i}\right). \tag{3.15}$$

Here, $\zeta(z)$ is the Weierstrass ζ-function which is a quasi-periodic function on E_τ:

$$\zeta(z + 2\pi i) = \zeta(z) + 2\zeta(\pi i)\,, \qquad \zeta(z + 2\pi i\tau) = \zeta(z) + 2\zeta(\pi i\tau)\,. \tag{3.16}$$

In the x-plane, P_0 is mapped to $x = \infty$. Note that x is multi-valued on $\Sigma_{\rm int}$ because, although it is periodic around the pre-images a_i of the a-cycle of the base torus E_τ, it picks up an additive constant im around the pre-images b_i of the b-cycle of the base torus. So restricting x to a single sheet there are N pairs of cuts $C_i^- = [k_i, l_i]$ and $C_i^+ = [k_i + im, l_i + im]$ which are identified. In this picture the genus N surface $\Sigma_{\rm int}$ is realized as the complex x-plane with N handles formed by identifying the top/bottom C_i^+ with the bottom/top of C_i^-. In general a Riemann surface of this form has $2N$ complex moduli which we can take to be the positions of the ends of the lower cuts $\{k_i, l_i\}$. We denote the moduli space of these surfaces as $\hat{\mathcal{M}}$. Clearly the moduli space of N-fold covers of the base torus E_τ, $\mathcal{M}_{\rm int}$, is only a complex N-dimensional subspace of this larger moduli space. Note that when the surface $\Sigma_{\rm int}$ degenerates, cuts in the x-plane merge. Note that two pairs of cuts can annihilate in two distinct ways. Either the bottom cut of one pair merges with the bottom cut of another pair to leave a single pair of the same kind, or the top cut of one pair merges with the bottom cut of another pair resulting in another pair of cuts now separated by $2im$ rather than im. By iterating this procedure we see that degenerations of $\Sigma_{\rm int}$ are manifested in the complex x-plane by pairs of cuts which join to form handles which can be separated by any integer multiple of im.

Before we discuss the matrix model approach let us consider the vacuum structure from the point-of-view of the tree-level superpotential:

$$W = \frac{1}{g_{YM}^2}{\rm Tr}\big(i\Phi[\Phi^+, \Phi^-] + m\Phi^+\Phi^- + W(\Phi)\big). \tag{3.17}$$

The case with a quadratic superpotential $W(\Phi)$ was considered originally in [13], here, we present the generalization for arbitrary polynomial functions $W(\Phi)$. At tree-level, the supersymmetric vacua can be found by solving the F-flatness conditions

$$[\Phi, \Phi^+] = im\Phi^+\,, \qquad [\Phi, \Phi^-] = -im\Phi^-\,, \qquad [\Phi^+, \Phi^-] = iW'(\Phi) \tag{3.18}$$

modulo complex gauge transformations. Using the symmetries we can diagonalize Φ and the build up solutions from a series of irreducible blocks. In such a block of size p $\{\Phi^\pm, \Phi\}$ have the same non-zero elements as $\{J^\pm, J_3\}$ of the irreducible representation of SU(2) of size p. In particular

$$\Phi = imJ_3 + \lambda\mathbf{1}\,. \tag{3.19}$$

The parameter λ is then determined by demanding

$${\rm Tr}_{\rm block}\,[\Phi^+, \Phi^-] = 0 = i\sum_{j=1}^{p} W'\Big(\lambda + \frac{im}{2}(p + 1 - 2j)\Big)\,. \tag{3.20}$$

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If $W(x)$ is a polynomial of degree n, there are $n-1$ possibilities for λ. Hence, a general vacuum corresponds to the data

$$\{n_j, p_j, \lambda_j,\ j = 1, \ldots, g\}\,, \qquad N = \sum_{j=1}^{g} n_j p_j \qquad (p_i \neq p_j \text{ when } \lambda_i = \lambda_j)\,, \tag{3.21}$$

where n_j is the number of blocks of size p_j associated to one of the $n-1$ roots λ_j of (3.20). The unbroken gauge group in this vacuum is

$$\prod_{j=1}^{g} \mathrm{U}(n_j) \tag{3.22}$$

and in particular the number of abelian factors is g. Quantum mechanically we expect in the infra-red that the non-abelian parts of the gauge group confine to leave an abelian theory with gauge group $\mathrm{U}(1)^g$. Note that the maximal value of g is N, obtained when the potential $W(x)$ is a polynomial of degree $> N$ and $p_j = n_j = 1$, $j = 1, \ldots, N$. In this case the unbroken gauge group is $\mathrm{U}(1)^N$. This is the $\mathcal{N} = 2^*$ analogue of the vacuum considered in [10] which can track the Coulomb branch of the $\mathcal{N} = 2^*$ theory and be used to extract the Donagi-Witten curve from the matrix model (as recently considered in [12]). The minimal value of $g = 1$ is obtained when $p_1 n_1 = N$. In other words when p_1 is an integer divisor of N. These are the "massive" vacua considered in [1, 20, 21] (note that in the $\mathrm{U}(N)$, as opposed to the $\mathrm{SU}(N)$ theory, there is an unbroken $\mathrm{U}(1)$ factor and so strictly speaking the vacua are not massive).

Now we turn to the matrix model approach to calculating the exact superpotential. According to Dijkgraaf and Vafa we consider a matrix model whose matrix variables are associated to the chiral superfields of the theory and whose action is the F-term. In the $\mathcal{N} = 2^* \to 1^*$ theory this yields a matrix model with a partition function (see [7, 20, 21, 22, 23, 24] for previous work on matrix models and the $\mathcal{N} = 1^*$ theory and its generalizations):

$$Z = \int [d\Phi^+][d\Phi^-][d\Phi] \exp -g_s^{-1} \operatorname{Tr}\left(i\Phi[\Phi^+, \Phi^-] + m\Phi^+\Phi^- + W(\Phi)\right). \tag{3.23}$$

Since $\Phi^\pm$ appear gaussian we may integrate them out:

$$Z = \int [d\Phi] \frac{e^{-g_s^{-1} \operatorname{Tr} V(\Phi)}}{\det(\mathrm{Adj}_\Phi + im)}\,. \tag{3.24}$$

In order to avoid confusion, we will suppose that the matrices have size $\hat{N}$. In order to implement the Dijkgraaf-Vafa procedure to the vacua described above, we need to solve the matrix model eq. (3.24) in the large $\hat{N}$-limit around the saddle-point corresponding to the classical solution for where Φ takes its tree-level form with n_i replaced by arbitrary variables $\hat{n}_i$ which individually tend to infinity. As usual in the large-$\hat{n}_i$ limit the eigenvalues of Φ spread out from their classical values along cuts on the complex eigenvalue x-plane. In other words, for each $j = 1, \ldots, g$ there will a set of p_j cuts which form a group, each element of which being the image of the lower one under translations ikm, $k = 1, \ldots, p_j - 1$. Each group is located in the vicinity of λ_j. So for each $j = 1, \ldots, g$ there are 2 complex parameters which one think of as the two ends of the lower cut. The density of eigenvalues $\rho(\phi)$ only has non-zero support along the cuts in the x-plane. Moreover, for a each $j = 1, \ldots, g$ the density of eigenvalues along each of the p_j cuts in the group is equal. The saddle-point equation in the large-N limit is most conveniently written in terms of the resolvent function

$$\omega(x) = \int_{\text{cuts}} \frac{\rho(\phi)}{x - \phi} d\phi \ , \qquad \int_{\text{cuts}} \rho(\phi) d\phi = 1 \ . \tag{3.25}$$

The resolvent $\omega(x)$ is an analytic function on the complex x-plane whose only singularities are branch cuts along the cuts where the eigenvalues are located for which the discontinuity across the cut gives the eigenvalue density

$$\omega(\phi + i\epsilon) - \omega(\phi - i\epsilon) = -2\pi i \rho(\phi) \, , \qquad \phi \in \text{cuts} \, . \tag{3.26}$$

In this, and following equations, ϵ is a suitable infinitesimal regulator. The saddle-point equation expresses the condition of zero force on a test eigenvalue in the presence of the large-N distribution of eigenvalues along cuts:

$$\frac{W'(\phi)}{S} = \omega(\phi + i\epsilon) + \omega(\phi - i\epsilon) - \omega(\phi + im) - \omega(\phi - im) \, , \qquad \phi \in \text{cuts} \, , \tag{3.27}$$

where $S = g_s \hat{N}$ is the 't Hooft coupling. This equation can be re-written in terms of the useful function

$$G(x) = U(x) + iS \left(\omega \left(x + \frac{im}{2} \right) - \omega \left(x - \frac{im}{2} \right) \right) , \tag{3.28}$$

where $U(x)$ is a polynomial in x of the same degree as $W(x)$ such that

$$W'(x) = -iU \left(x + \frac{im}{2} \right) + iU \left(x - \frac{im}{2} \right) . \tag{3.29}$$

It turns out that $G(x)$ has a somewhat simpler analytic structure than $\omega(x)$. For a given group of p_j cuts the situation is described in [21]. In taking the difference of the resolvents in (3.28) most of the cuts cancel to leave only a pair of cuts associated to each group $[k_j, l_j]$ and $[k_j + imp_j, l_j + imp_j]$. Moreover the saddle-point equations simply imply a gluing condition between each pair of cuts where the bottom/top of the lower cut is identified with the top/bottom of the upper cut. This naturally defines a Riemann surface of genus g since each pair of cuts once glued corresponds to one handle. In fact it should not escaped the reader's notice that this Riemann surface is precisely an example of the family of surfaces $\hat{\mathcal{M}}$ defined earlier in the context of the integrable system.

However there is more structure since the Riemann surface has to admit a meromorphic function G with a singularity at the point P_0, $x(P_0) = \infty$, with the following pole structure

$$\lim_{P \to P_0} G(P) = U(x) + \mathcal{O} \left(\frac{1}{x} \right) . \tag{3.30}$$

So the saddle-point equation of the matrix model boils down to the existence of a genus g Riemann surface Σ_{mm} in the space $\hat{\mathcal{M}}$ which admits a meromorphic function G whose only singularity is at the point P_0 with the specific pole structure (3.30).

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The question is how many moduli does the surface $\Sigma_{\rm mm}$ have? Potentially there are $2g$ complex moduli as pointed out above. However, there are additional constraints arising from the requirement that the function G exists on $\Sigma_{\rm mm}$. Generically, by the Riemann-Roch theorem, such a meromorphic function only exist on a surface of genus $g < n$, in which case there would be $n - g$ non-trivial meromorphic functions with a pole of this order or less. Therefore prescribing the asymptotic form (3.30) amounts to g non-trivial conditions of the surface $\Sigma_{\rm mm}$, leaving a g complex dimensional subspace of moduli. The g moduli of these surfaces can be described by the g quantities

$$S_i = -\frac{1}{2\pi}\oint_{A_i} G(x)dx\,, \tag{3.31}$$

where A_i is a contour which encircles the lower cut $[k_i, l_j]$ of each pair. Notice that the number of matrix model eigenvalues associated to each $j = 1, \ldots, g$ is equal to $p_j S_j/g_s$ and furthermore

$$S_j = g_s\hat{n}_j\,, \qquad S = g_s\hat{N} = \sum_{j=1}^{g} p_j S_j\,. \tag{3.32}$$

Each of the moduli S_j will become a field of the Dijkgraaf-Vafa superpotential identified with the glueball superfield of the unbroken $\mathrm{U}(n_j)$ factor of the gauge group.

The second ingredient required to construct the Dijkgraaf-Vafa superpotential is the variation of the genus zero free energy $\mathcal{F}_0$ with respect to S_j. Following [21], this is equal to

$$\frac{\partial\mathcal{F}_0}{\partial S_j} = -i\oint_{B_j} G(x)dx\,, \tag{3.33}$$

where B_j is the conjugate cycle to A_j, in other words, goes from the lower cut to the upper cut of a pair. The quantity (3.33) can be interpreted physically as the variation of the genus zero free energy of the matrix model in transporting p_j eigenvalues in from infinity and placing one on each of the p_j cuts in a group (so as to maintain the same density along each of the cuts in the group). We conjecture that the generalization of the Dijkgraaf-Vafa glueball superpotential for the vacuum in question is then

$$W_{\rm eff}(S_j) = \sum_{j=1}^{g}\Big(n_j\frac{\partial\mathcal{F}_0}{\partial S_j} - 2\pi i p_j\tau S_j\Big)\,, \tag{3.34}$$

where now the n_j are the physical rather than matrix model quantities.

A critical point of $W_{\rm eff}$ corresponds to

$$\sum_{j=1}^{g} n_j\frac{\partial^2\mathcal{F}_0}{\partial S_k\partial S_j} = 2\pi i\tau p_k\,. \tag{3.35}$$

This equation can be written in a more suggestive way by noticing that

$$\omega_j = -\frac{1}{2\pi}\frac{\partial}{\partial S_j}G(x)dx\,, \tag{3.36}$$

are a basis for the g holomorphic 1-forms on $\Sigma_{\rm mm}$ since the singular part of $G(x)dx$ at $x=\infty$ is manifestly independent of the moduli $\{S_j\}$. Furthermore, the ω_i are normalized so that

$$\oint_{a_j} \omega_k = \delta_{jk}\,. \tag{3.37}$$

Hence

$$\frac{\partial^2\mathcal{F}_0}{\partial S_k \partial S_j} = 2\pi i \oint_{B_j} \omega_k = 2\pi i \tau_{jk}\,, \tag{3.38}$$

where τ_{jk} is the period matrix of $\Sigma_{\rm mm}$. Consequently the critical point equations are

$$\sum_{j=1}^{g} n_j \tau_{jk} = \tau p_k\,. \tag{3.39}$$

These equations are precisely the conditions that $\Sigma_{\rm mm}$ is an N-sheeted covering of the base torus E_τ. In order to prove this we need to find a map from $\Sigma_{\rm mm}$ to E_τ which covers the latter N times. For $P \in \Sigma_{\rm mm}$, the map is simply

$$z(P) = 2\pi i \int_{P'}^{P} \sum_{j=1}^{g} n_j \omega_j \quad \mathrm{mod}\ 2\pi i, 2\pi i \tau\,, \tag{3.40}$$

where P' is a fixed but arbitrarily chosen point of $\Sigma_{\rm mm}$. Since $N = \sum_j n_j p_j$ this map covers E_τ N times. In particular, z is identified with the same quantity on the integrable system side. Similar ideas have been expressed in [12] for the case when $g = N$, $n_j = p_j = 1$ corresponding to the vacuum with unbroken $\mathrm{U}(1)^N$ symmetry.

Now we compare the integrable system with the matrix model. The first point is that the moduli space $\mathcal{M}_{\rm int}$ and the union of the moduli spaces of the matrix model for different vacua $\mathcal{M}_{\rm mm}$ are both subspaces of the moduli space $\hat{\mathcal{M}}$ defined earlier. The subspace $\mathcal{M}_{\rm int}$ contains the surfaces which are N-fold branched coverings of the torus E_τ and the subspace $\mathcal{M}_{\rm mm}$ contains surfaces which admit a meromorphic function G with a fixed pole structure at P_0 determined by the $\mathcal{N}=1$ deformation. The critical-point condition in the integrable system approach boils down to the existence of the meromorphic function G while on the matrix model side it boils down to the constraint of being an N-fold cover of the torus E_τ. In other words, the vacua correspond to points of intersection between $\mathcal{M}_{\rm int}$ and the different components in $\mathcal{M}_{\rm mm}$. In this picture we must identify the polynomial $U(x)$ on both sides of the story which then implicitly determines the relation between $\tilde{W}(x)$ and $W(x)$. Notice for the case of the vacuum with $g = N$ and unbroken $\mathrm{U}(1)^N$ we can, generalizing the situation in the basic $\mathcal{N}=2$ theory [10], extract the Donagi-Witten curve from the matrix model model as was pointed out in [12].

Now we turn to the question of the value of the superpotential at the critical points calculated using the two methods. We start by finding a more explicit expression for the critical value of the superpotential on the matrix model side by the following manipulations:

$$W_{\rm eff} = -i \sum_{j=1}^{g} \left(n_j \oint_{B_j} G(x)\,dx - \tau p_j \oint_{a_j} G(x)\,dx \right)$$

$$
\begin{aligned}
&= -i\sum_{j=1}^{g} n_j\Big(\oint_{B_j} G(x)dx - \sum_{k=1}^{g}\tau_{jk}\oint_{A_k} G(x)\,dx\Big) \\
&= -2\pi\,\mathrm{Res}_{P_0}\big(U(x)z\,dx\big)\,. \qquad (3.41)
\end{aligned}
$$

In the above, to reach the second line we used the critical-point equations (3.39) and to reach the final line we applied a Riemann bilinear relation, used the fact that $dz = \sum_j n_j\omega_j$ and that $G(x)$ can be replaced with $U(x)$ in the vicinity of P_0. A similar expression was derived in [12].

Now from the integrable system side. Once again we use the fact that action variables conjugate to the angle are integrals of the Seiberg-Witten differential:

$$
a_j = \oint_{A_j}\lambda\,, \qquad \lambda = v\,dz\,. \qquad (3.42)
$$

Since the residue of λ at P_0 is independent of the moduli a_i, the derivative of λ with respect to a_i is the holomorphic 1-form ω_i. However, with the Seiberg-Witten differential in the form $v\,dz$ we must be careful to differentiate at fixed z otherwise singularities arise:

$$
\omega_i = \Big(\frac{\partial v}{\partial a_i}\Big)_z dz\,. \qquad (3.43)
$$

It turns out that this is not a conventient form. One can eaily verify from the fact that $F(v,z;a_i)=0$ and from the relation between x and v and z, that

$$
\omega_i = \Big(\frac{\partial v}{\partial a_i}\Big)_z dz = -\Big(\frac{\partial z}{\partial a_i}\Big)_v dv = -\Big(\frac{\partial z}{\partial a_i}\Big)_x dx\,. \qquad (3.44)
$$

The final expression here is the one which will be most convenient. Applying a Riemann bilinear relation to Ω and ω_i one arrives at the following expression for the angular velocities

$$
\varpi_j = -2\pi i\frac{\partial}{\partial a_j}\mathrm{Res}_{P_0}\big(U(x)z\,dx\big)\,. \qquad (3.45)
$$

Hence the hamiltonian which degenerates the flow described by the abelian differential Ω is

$$
H = -2\pi\,\mathrm{Res}_{P_0}\big(U(x)z\,dx\big) \qquad (3.46)
$$

which is in perfect agreement with the matrix model result (3.41). In a sequel we shall show how to write (3.46) in terms of the Lax matrix of the integrable system and hence has a function of the positions and momenta.

Obviously important questions remain, the most interesting concerning the exceptional vacua for which the surface has a genus greater than or equal to the order of $W(x)$ and for which P_0 must be a Weierstrass point.

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References

[1] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **07** (1999) 021 [hep-th/9906011].

[2] N. Dorey, T.J. Hollowood and S. Prem Kumar, *An exact elliptic superpotential for $N = 1^*$ deformations of finite $N = 2$ gauge theories*, *Nucl. Phys.* **B 624** (2002) 95 [hep-th/0108221].

[3] T.J. Hollowood and S.P. Kumar, *World sheet instantons via the myers effect and $N = 1^*$ quiver superpotentials*, *J. High Energy Phys.* **10** (2002) 077 [hep-th/0206051].

[4] N. Dorey and A. Sinkovics, *$N = 1^*$ vacua, fuzzy spheres and integrable systems*, *J. High Energy Phys.* **07** (2002) 032 [hep-th/0205151].

[5] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[6] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[7] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[8] R. Dijkgraaf and C. Vafa, *$N = 1$ supersymmetry, deconstruction and bosonic gauge theories*, hep-th/0302011.

[9] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[10] F. Cachazo and C. Vafa, *$N = 1$ and $N = 2$ geometry from fluxes*, hep-th/0206017.

[11] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, *Nonperturbative superpotentials and compactification to three dimensions*, hep-th/0304061.

[12] M. Petrini, A. Tomasiello and A. Zaffaroni, *On the geometry of matrix models for $N = 1$*, *J. High Energy Phys.* **08** (2003) 004 [hep-th/0304251].

[13] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101].

[14] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, *Phys. Rept.* **71** (1981) 313.

[15] H.M. Farkas and I. Kra, *Riemann Surfaces*, Springer Verlag, New York 1980.

[16] E.J. Martinec, *Integrable structures in supersymmetric gauge and string theory*, *Phys. Lett.* **B 367** (1996) 91 [hep-th/9510204].

[17] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[18] I.M. Krichever, *Funct. Annal. Appl.* **14** (1980) 45.

[19] I. Krichever, O. Babelon, E. Billey and M. Talon, *Spin generalization of the calogero-moser system and the matrix kp equation*, hep-th/9411160.

[20] N. Dorey, T.J. Hollowood, S. Prem Kumar and A. Sinkovics, *Exact superpotentials from matrix models*, *J. High Energy Phys.* **11** (2002) 039 [hep-th/0209089].

[21] N. Dorey, T.J. Hollowood, S.P. Kumar and A. Sinkovics, *Massive vacua of $N = 1^*$ theory and S-duality from matrix models*, *J. High Energy Phys.* **11** (2002) 040 [hep-th/0209099].

[22] N. Dorey, T.J. Hollowood and S.P. Kumar, *S-duality of the Leigh-Strassler deformation via matrix models*, *J. High Energy Phys.* **12** (2002) 003 [hep-th/0210239].

[23] T.J. Hollowood, *New results from glueball superpotentials and matrix models: the Leigh-Strassler deformation*, *J. High Energy Phys.* **04** (2003) 025 [hep-th/0212065].

[24] T.J. Hollowood, *Five-dimensional gauge theories and quantum mechanical matrix models*, *J. High Energy Phys.* **03** (2003) 039 [hep-th/0302165].

[25] I.M. Krichever and D.H. Phong, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories*, *J. Diff. Geom.* **45** (1997) 349 [hep-th/9604199].

[26] I.M. Krichever and D.H. Phong, *Symplectic forms in the theory of solitons*, hep-th/9708170.

[27] E. D'Hoker, I.M. Krichever and D.H. Phong, *Seiberg-Witten theory, symplectic forms and hamiltonian theory of solitons*, hep-th/0212313.

[28] A. Gorsky and N. Nekrasov, *Elliptic Calogero-Moser system from two-dimensional current algebra*, hep-th/9401021.

[29] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple singularities and $N = 2$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 344** (1995) 169 [hep-th/9411048].

[30] P.C. Argyres and A.E. Faraggi, *The vacuum structure and spectrum of $N = 2$ supersymmetric* $\mathrm{SU}(N)$ *gauge theory*, *Phys. Rev. Lett.* **74** (1995) 3931 [hep-th/9411057].

[31] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[32] F. Cachazo, N. Seiberg and E. Witten, *Chiral rings and phases of supersymmetric gauge theories*, *J. High Energy Phys.* **04** (2003) 018 [hep-th/0303207].